Group Theory Resit Exam

Date: 4 February 2022

Place: Aletta Jacobs Hall 1

Time: 08:30 – 10:30

Instructions

- To get full points, you must provide complete arguments and computations. You will get no points if you do not explain your answer. Answers like "True", "False", "34", "Yes", "No" will not be accepted.
- While solving a problem, you can use any statement that needs to be proven as a part of another problem even if you did not manage to prove it; e.g. you can use part (a) while solving part (b) even if you did not prove (a).
- Clearly write your name and student number on each page you submit.
- The examination consists of 5 questions. You can score up to 36 points and you get 4 points for free. This way you will score in total between 4 and 40 points.

Problems

1. Let S_{10} be the permutation group on $\{1,\ 2,\ \ldots,\ 9,\ 10\}$.
   (a) **[2 pt]** Compute the order of $(2\ 5\ 7\ 8\ 4)^{121}(5\ 1\ 3)^{26}(2\ 8)$ in S_{10}.
   (b) **[3 pt]** Give a representative for each conjugacy class of elements of order 4 in S_{10}.
   (c) **[3 pt]** Find a set of generators of a subgroup in S_{10} that is isomorphic to $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}$.
2. Let H, K and N be groups and let $f : H \to K$ and $g : K \to N$ be homomorphisms.
   (a) **[2 pt]** Show that the composition $g \circ f : H \to N$ is a homomorphism.
   (b) **[3 pt]** Show that the kernel $\ker(f)$ is a normal subgroup of $\ker(g \circ f)$.
   (c) **[3 pt]** Suppose that the order of H is a prime p and suppose also that $g \circ f$ is injective. Prove that there is an element in K of order p.
3. Let G be a group and let p be a prime dividing the order of G. Let $\mathrm{Syl}(p)$ be the set of Sylow p-groups of G.
   (a) **[2 pt]** Show that the map
   $$\begin{aligned} G \times \mathrm{Syl}(p) &\to \mathrm{Syl}(p), \\ (g, P) &\mapsto g \cdot P := gPg^{-1} \end{aligned}$$
   is a group action.
   (b) **[2 pt]** Is the action in (a) transitive? Explain.
   (c) **[3 pt]** Let $S_{\mathrm{Syl}(p)}$ denote the symmetric group of $\mathrm{Syl}(p)$. Show that the image of the homomorphism
   $$f : G \to S_{\mathrm{Syl}(p)},$$
   given by $f(g)(P) = g \cdot P$, is trivial if and only if there is only one Sylow p-group of G.

4. **[4 pt]** List, up to isomorphism, all abelian groups of order at most 50, containing at least 3 elements of order 2 and at least 1 element of order 3.

5. **Prove/Disprove.** For each of the following statements, prove the statement if it is true and disprove it if it is false.

   (a) **[3 pt]** A group is abelian if and only if every conjugacy class contains exactly one element.

   (b) **[3 pt]** There is a surjective homomorphism from S_6 to $\mathbb{Z}/77\mathbb{Z}$.

   (c) **[3 pt]** If a group G of order 15 acts on a set X with 24 elements then it is possible for X to contain only 3 orbits.

GOOD LUCK! ☺